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INTEGRALS WITH A HIDDEN SOLUTION: NO NEED FOR COMPLEX CALCULATIONS

Abstract. *This article presents a collection of definite integrals that, while seemingly beyond the reach of standard high school techniques, can be evaluated using elementary properties and the Newton–Leibniz formula. These examples are designed to shift students’ perception of integration from a purely procedural task to a conceptual exploration. The methods and insights discussed here are intended to enrich classroom instruction and serve as compact, curriculum-aligned challenge problems.*

Key words: *Definite integrals; Mathematics education; Conceptual exploration; Challenge problems.*

1. Introduction

The definite integral is typically introduced to high school students as a computational tool, evaluated through antiderivatives and standard algebraic techniques. However, many integrals that appear intractable using these methods can be resolved with surprisingly simple ideas already within students’ reach such as symmetry, properties of even and odd functions, and the Newton-Leibniz formula.

This article aims to demonstrate how elegant reasoning can unlock integrals that initially seem complex or inaccessible. By presenting problems whose solutions rely not on advanced techniques but on insightful application of basic principles, we seek to highlight the conceptual beauty of integration. In doing so, we hope to foster mathematical curiosity and provide teachers with meaningful examples to enrich classroom learning.

The integrals presented here can be used as “short questions” in the high school curriculum, particularly in advanced-level courses. They emphasize not only technical skill but also strategic thinking an essential component of mathematical education.

2. Explanations and Justifications

The concept of the integral is not only fundamental in school-level mathematics, but also essential for further academic studies. According to research by (Jones, 2014), most

students perceive a definite integral either as the area under a curve or as an 'antiderivative', while only a minority recognize it as a sum. The findings of (Serhan, 2015) suggest that although students are capable of applying procedural knowledge to solve integration tasks, they often lack a deeper conceptual understanding of basic integration concepts.

In preparing this article, we asked ourselves how we could introduce students to integrals that do not extend beyond the boundaries of the high school curriculum. Of course, there are classical textbooks such as Thomas' Calculus (Thomas et al., 2004), upon which many high school materials are based. These typically present definite integrals as computational exercises reliant on indefinite integrals.

Our focus, instead, is on properties that follow from the definition of the definite integral and its inherent characteristics within the school curriculum. An additional objective is to generate a sense of mathematical discovery by revealing seemingly 'hidden' solutions and providing students with a meaningful experience of mathematical insight (Braverman, Samovol, Applebaum, 2004). Students also gain a valuable takeaway: even when an indefinite integral cannot be found for a specific function, the corresponding definite integral may still be computable.

3. The Newton–Leibniz Formula as a Source of Interesting Properties

We begin by presenting several calculus definitions and theorems we will use (Thomas et al., 2004).

Definition 1. *A function F is an antiderivative of f on an interval D if for all x in D*

$$F'(x) = f(x).$$

The Fundamental Theorem of Calculus, Part 1.

Theorem 1. *If f is continuous on $[a, b]$, then*

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$, differentiable on (a, b) , and its derivative is $f(x)$; that is, $F'(x) = f(x)$.

The Fundamental Theorem of Calculus, Part 2 (Newton–Leibniz Formula).

Theorem 2. *If f is continuous at every point of $[a, b]$, and F is an antiderivative of f over $[a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a). \quad (*)$$

It is also worth noting that from the derivative property $(F(kx + n))' = kF'(kx + n)$ and from formula (*), it follows that:

$$\int_a^b f(kx + n)dx = \frac{1}{k} F(kx + n) \Big|_a^b = \frac{1}{k} F(kb + n) - \frac{1}{k} F(ka + n). \quad (**)$$

Let $f(x)$ be a function defined and continuous for all $a \leq x \leq b$. We will now demonstrate several important properties of definite integrals that can be derived using the Newton–Leibniz formula (*).

$$\textbf{Property (1)} \quad I_1 = \int_a^b f(x)dx = \int_a^b f(a + b - x)dx.$$

$$\textbf{Property (2)} \quad I_2 = \int_{-\frac{(b-a)}{2}}^{\frac{(b-a)}{2}} f\left(\frac{a+b}{2} - x\right)dx = \int_a^b f(x)dx.$$

If $f(x)$ is defined and continuous for all $0 \leq x \leq 1$, then:

$$\textbf{Property (3)} \quad I_3 = \int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx.$$

If $f(x)$ is defined and continuous for all $-a \leq x \leq a$, then:

$$\textbf{Property (4)} \quad I_4 = \int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx.$$

If $f(x)$ is defined on the interval $[-a, a]$ and is an even function then:

$$\textbf{Property (5)} \quad I_5 = \int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx.$$

If $f(x)$ is defined on the interval $[-a, a]$ and is an odd function then:

Property (6) $I_6 = \int_{-a}^a f(x)dx = 0.$

Note: Before each full solution, we offer a brief conceptual clue. Readers who grasp the underlying idea may choose to skip the detailed proof.

3.1. Proving the properties

Note: In all cases, F is an antiderivative of the function f .

Clues: If the function $f(x)$ is defined and continuous for all $a \leq x \leq b$ then:

$$I_1 = \int_a^b f(x)dx = F(b) - F(a).$$

1. Now prove that:

$$I_1 = \int_a^b f(a + b - x) dx.$$

2. Prove **Property (2)** using the Newton-Leibniz Formula.
3. Prove **Property (3)** using **Property (1)** for $a = 0$, $b = 0,5 \cdot \pi$.
4. Prove **Property (4)** using the Newton-Leibniz Formula.
5. Prove **Property (5)** using the Newton-Leibniz Formula.
6. Prove **Property (6)** using the Newton-Leibniz Formula.

Full proof.

To prove Properties (1) – (6), we will use the Newton–Leibniz Formula:

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a) \quad (*)$$

and also, the property:

$$\int_a^b f(kx + n)dx = \frac{1}{k} F(kx + n) \Big|_a^b = \frac{1}{k} F(kb + n) - \frac{1}{k} F(ka + n). \quad (**)$$

We will now prove the properties:

Property (1) follows from formulas (*) and (**):

$$\begin{aligned}
 I_1 &= \int_a^b f(a+b-x)dx = -F(a+b-x) \Big|_a^b = -F(a+b-b) + F(a+b-a) = \\
 &= F(b) - F(a) = \int_a^b f(x)dx.
 \end{aligned} \tag{1}$$

Property (2) is established using (*) and (**) as well:

$$\begin{aligned}
 I_2 &= \int_{-\frac{(b-a)}{2}}^{\frac{(b-a)}{2}} f\left(\frac{a+b}{2}-x\right)dx = -F\left(\frac{a+b}{2}-x\right) \Big|_{-\frac{(b-a)}{2}}^{\frac{(b-a)}{2}} = \\
 &= -F\left(\frac{a+b}{2}-\frac{b-a}{2}\right) + F\left(\frac{a+b}{2}+\frac{b-a}{2}\right) = \\
 &= F(b) - F(a) = \int_a^b f(x)dx.
 \end{aligned} \tag{2}$$

Property (3) proof we will use **Property (1)**:

$$I_3 = \int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f\left(\sin\left(0+\frac{\pi}{2}-x\right)\right)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx. \tag{3}$$

Property (4) is established as follows:

$$I_4 = \int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx. \tag{4}$$

Applying **Property (1)** yields:

$$\int_{-a}^0 f(x)dx + \int_0^a f(x)dx = \int_{-a}^0 f(-a+0-x)dx + \int_0^a f(x)dx.$$

According to (**) we obtain:

$$\int_{-a}^0 f(-a+0-x)dx + \int_0^a f(x)dx = -F(-a-x) \Big|_{-a}^0 + \int_0^a f(x)dx =$$

$$= F(0) - F(-a) + \int_0^a f(x)dx.$$

We proceed according to (**):

$$F(0) - F(-a) + \int_0^a f(x)dx = -(F(-a) - F(-0)) + \int_0^a f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx.$$

Property (5). First, **Property (4)** is applied:

$$I_5 = \int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx.$$

Given the fact that the function is an even function we obtain:

$$\int_0^a f(x)dx + \int_0^a f(-x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2 \int_0^a f(x)dx.$$

Property (6). Again, begin with **Property (4)**:

$$I_6 = \int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx.$$

Then we will use the fact that the function is odd and obtain:

$$\int_0^a f(x)dx + \int_0^a f(-x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0.$$

3.1. Examples

Example 1. Evaluate the following integral using **Property (1)**:

$$I = \int_2^4 \frac{\ln x^2}{\ln x^2 + \ln(x^2 - 12x + 36)} dx.$$

(This example is taken from the 48th William Lowell Putnam Competition, 1987).

Solving. Clues:

- 1) Prove using **Property (1)** (with $a = 2$, $b = 4$) that:

$$I = \int_2^4 \frac{\ln x}{\ln x + \ln(6-x)} dx.$$

2) Prove that:

$$I = \int_2^4 \frac{\ln(6-x)}{\ln(6-x) + \ln x} dx.$$

3) Prove that:

$$2I = \int_2^4 \frac{\ln x + \ln(6-x)}{\ln x + \ln(6-x)} dx = 2.$$

Calculate I in a different way.

Full proof. First, simplify the expression inside the integral using logarithmic properties:

$$\begin{aligned} I &= \int_2^4 \frac{\ln x^2}{\ln x^2 + \ln(x^2 - 12x + 36)} dx = \\ &= \int_2^4 \frac{2 \ln x}{2 \ln x + \ln(x-6)^2} dx = \int_2^4 \frac{2 \ln x}{2 \ln x + 2 \ln(6-x)} dx = \int_2^4 \frac{\ln x}{\ln x + \ln(6-x)} dx. \end{aligned}$$

Now apply **Property (1)**:

$$\begin{aligned} &\int_2^4 \frac{\ln x}{\ln x + \ln(6-x)} dx = \\ &= \int_2^4 \frac{\ln(2+4-x)}{\ln(2+4-x) + \ln(6-(2+4-x))} dx = \int_2^4 \frac{\ln(6-x)}{\ln(6-x) + \ln x} dx. \end{aligned}$$

Then:

$$I = \int_2^4 \frac{\ln x}{\ln x + \ln(6-x)} dx.$$

But also:

$$I = \int_2^4 \frac{\ln(6-x)}{\ln(6-x) + \ln x} dx.$$

Therefore:

$$I + I = \int_2^4 \frac{\ln x}{\ln x + \ln(6-x)} dx + \int_2^4 \frac{\ln(6-x)}{\ln x + \ln(6-x)} dx = \int_2^4 dx = 2.$$

And in conclusion:

$$2I = 2 \Rightarrow I = 1.$$

Answer: $I = 1$.

Example 2. Evaluate the following integral using **Property (3)**:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx.$$

Solving. Apply **Property (3)**:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx.$$

Add the two resulting integrals:

$$I + I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx.$$

Therefore:

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$$

So:

$$I = \frac{\pi}{4}.$$

Answer: $I = \frac{\pi}{4}$.

Example 3. Evaluate the following integral using **Properties (2)** and **(4)**

$$I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}} dx.$$

Solving. Clues:

1. Prove using **Property (2)** with $a = 2$, $b = 4$ that:

$$I = \int_{-1}^1 \frac{\sqrt{\ln(6+x)}}{\sqrt{\ln(6-x)} + \sqrt{\ln(9-x)}} dx.$$

2. Prove using **Property (4)** with $a = -1$, $b = 1$ that:

$$I = \int_0^1 \frac{\sqrt{\ln(6+x)} + \sqrt{\ln(6-x)}}{\sqrt{\ln(6-x)} + \sqrt{\ln(6+x)}} dx.$$

Full proof. Apply **Property (2)**:

$$\begin{aligned} I &= \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}} dx = \\ &= \int_{-\frac{4-2}{2}}^{\frac{4-2}{2}} \frac{\sqrt{\ln\left(9 - \left(\frac{4+2}{2} - x\right)\right)}}{\sqrt{\ln\left(\left(\frac{4+2}{2} - x\right) + 3\right)} + \sqrt{\ln\left(9 - \left(\frac{4+2}{2} - x\right)\right)}} dx = \\ &= \int_{-1}^1 \frac{\sqrt{\ln(6+x)}}{\sqrt{\ln(6-x)} + \sqrt{\ln(6+x)}} dx. \end{aligned}$$

Then use **Property (4)**:

$$\begin{aligned} &\int_{-1}^1 \frac{\sqrt{\ln(6+x)}}{\sqrt{\ln(6-x)} + \sqrt{\ln(6+x)}} dx = \\ &= \int_0^1 \frac{\sqrt{\ln(6+x)}}{\sqrt{\ln(6-x)} + \sqrt{\ln(6+x)}} dx + \int_0^1 \frac{\sqrt{\ln(6-x)}}{\sqrt{\ln(6+x)} + \sqrt{\ln(6-x)}} dx = \end{aligned}$$

$$= \int_0^1 \frac{\sqrt{\ln(6+x)} + \sqrt{\ln(6-x)}}{\sqrt{\ln(6-x)} + \sqrt{\ln(6+x)}} dx = \int_0^1 dx = 1.$$

Answer: $I = 1$.

Example 4. Evaluate the following integral using **Properties (5)** and **(6)**:

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^7 - 5x^5 - 3x^3 + 8 \tan^2 x}{\cos^2 x} dx.$$

Solving. Clues:

1. Prove that:

$$I_1 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^7 - 5x^5 - 3x^3}{\cos^2 x} dx = 0.$$

2. Prove that:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{8 \tan^2 x}{\cos^2 x} dx = 8 \cdot \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x d(\tan x).$$

3. Calculate that $I = I_1 + I_2$.

Full proof. Rewrite the integral:

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^7 - 5x^5 - 3x^3 + 8 \tan^2 x}{\cos^2 x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^7 - 5x^5 - 3x^3}{\cos^2 x} dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{8 \tan^2 x}{\cos^2 x} dx.$$

The function

$$f(x) = \frac{x^7 - 5x^5 - 3x^3}{\cos^2 x}$$

is odd so:

$$f(-x) = \frac{(-x)^7 - 5(-x)^5 - 3(-x)^3}{\cos^2(-x)} = -\frac{x^7 - 5x^5 - 3x^3}{\cos^2 x} = -f(x).$$

The function

$$g(x) = \frac{8 \tan^2 x}{\cos^2 x}$$

is even so:

$$g(-x) = \frac{8 \tan^2(-x)}{\cos^2(-x)} = \frac{8 \tan^2 x}{\cos^2 x} = g(x).$$

Therefore, by **Properties (5) and (6)**:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^7 - 5x^5 - 3x^3}{\cos^2 x} dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{8 \tan^2 x}{\cos^2 x} dx = 0 + 2 \int_0^{\frac{\pi}{4}} 8 \tan^2 x d(\tan x) = \frac{8 \tan^3 x}{3} \Big|_0^{\frac{\pi}{4}} = \frac{8}{3}.$$

Answer: $I = \frac{8}{3}$.

Example 5. Evaluate the following:

$$\int_0^1 |f(x) - x| dx.$$

If you know that $f(x)$ is continuous and symmetric with respect to $y = x$ and $f(0) = 1$.

Solving. Clues:

1. Sketch $f(x)$ for $0 \leq x \leq 1$.
2. Look at the provided sketch of a general $f(x)$ (Figure 1).
3. Explain why $S_2 = S_3$, and $S_1 = S_4$.
4. Check if necessarily $S_2 + S_4 = 0,5$.
5. Prove that:

$$I = \int_0^1 |f(x) - x| dx = \int_0^{x_0} [f(x) - x] dx + \int_{x_0}^1 [x - f(x)] dx.$$

6. Prove that $I = 0,5 \cdot S_{ABCD}$.

7. How might this problem be generalized?

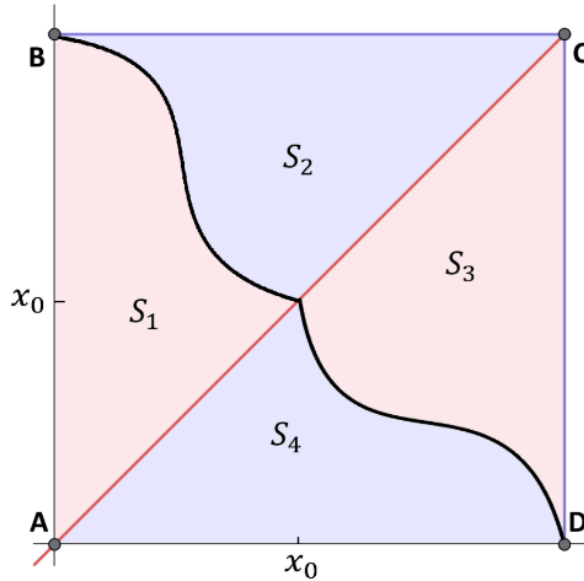


Figure 1. Sketch of $f(x)$.

Full proof. In Figure 1, point A is the origin, $B(0,1)$, $C(1,1)$, $D(1,0)$.

We know that $f(0) = 1$ and also that $f(x)$ is symmetrical with respect to $y = x$, it is its own inverse. Then $f(1) = 0$.

The function $f(x)$ intersects the line $y = x$ at the point (x_0, x_0) , since it is continuous, point B lies above the line, and point D lies below it. We therefore obtain: $f(x_0) = x_0$.

ABCD is a square with side length 1 the diagonal AC divides it into two congruent triangles, each of area $\frac{1}{2}$. Since $f(x)$ is symmetric with respect to $y = x$, we obtain: $S_2 = S_3$, and $S_1 = S_4$.

We can see that:

$$S_3 + S_4 = S_1 + S_2 = \frac{1}{2} S_{ABCD} = \frac{1}{2}.$$

Therefore:

$$S_2 + S_4 = S_3 + S_4 = \frac{1}{2}.$$

$$I = \int_0^1 |f(x) - x| dx = \int_0^{x_0} |f(x) - x| dx + \int_{x_0}^1 |f(x) - x| dx.$$

That is because x_0 is the intersection with $y = x$, if $x \geq x_0$ we get that $x \leq f(x)$. If $x \leq x_0$ we get that $x \geq f(x)$.

So, for $x \leq x_0$:

$$|f(x) - x| = f(x) - x$$

and for $x \geq x_0$:

$$|f(x) - x| = x - f(x).$$

$$I = \int_0^{x_0} [f(x) - x]dx + \int_{x_0}^1 [x - f(x)]dx = S_1 + S_3 = S_1 + S_2 = \frac{1}{2}.$$

$$I = \frac{1}{2}.$$

Answer: $I = \frac{1}{2}$.

3.3. Integrals with reversed limits of integration

Example 6. Evaluate the following integral:

$$I = \int_{\frac{1}{a}}^a \frac{\arctan x}{1 - x + x^2} dx, \quad a > 1.$$

Solving. Clues:

1. Prove that:

$$I = \int_a^{\frac{1}{a}} \frac{\arctan \frac{1}{t}}{t^2 - t + 1} dt.$$

2. Use the fact that:

$$\int_a^b f(x)dx = - \int_b^a f(x)dx.$$

3. Prove that:

$$I = \int_{\frac{1}{a}}^a \frac{\arctan \frac{1}{x}}{x^2 - x + 1} dx.$$

4. Calculate $2I$.

Full proof. This integral has limits of integration that are symmetric with respect to a transformation. Thus, we use the substitution $t = \frac{1}{x}$ which allows us to preserve the limits of integration.

We get $t = \frac{1}{x}$, $dt = -\frac{dx}{x^2}$ and if $x = \frac{1}{a}$, then $t = a$, if $x = a$ then $t = \frac{1}{a}$ ($a > 1$):

$$I = \int_{\frac{1}{a}}^a \frac{\arctan x}{1-x+x^2} dx = \int_a^{\frac{1}{a}} \frac{\arctan \frac{1}{t}}{1-\frac{1}{t}+\frac{1}{t^2}} \cdot \left(-\frac{dt}{t^2}\right) = -\int_a^{\frac{1}{a}} \frac{\arctan \frac{1}{t}}{t^2-t+1} dt.$$

We know that:

$$\int_a^b f(x) dx = -\int_b^a f(x) dx.$$

Therefore:

$$I = -\int_a^{\frac{1}{a}} \frac{\arctan \frac{1}{t}}{t^2-t+1} dt = \int_{\frac{1}{a}}^a \frac{\arctan \frac{1}{t}}{t^2-t+1} dt = \int_{\frac{1}{a}}^a \frac{\arctan \frac{1}{x}}{x^2-x+1} dx.$$

We now add the original integral to the expression we just obtained (which is equal to it):

$$I + I = \int_{\frac{1}{a}}^a \frac{\arctan x}{1-x+x^2} dx + \int_{\frac{1}{a}}^a \frac{\arctan \frac{1}{x}}{x^2-x+1} dx = \int_{\frac{1}{a}}^a \frac{\arctan x + \arctan \frac{1}{x}}{1-x+x^2} dx.$$

We will use the fact that for all real x , $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ therefore:

$$2I = \int_{\frac{1}{a}}^a \frac{\arctan x + \arctan \frac{1}{x}}{1-x+x^2} dx = \int_{\frac{1}{a}}^a \frac{\frac{\pi}{2}}{1-x+x^2} dx.$$

Complete the square in the denominator where appropriate, and simplify:

$$\begin{aligned}
 2I &= \frac{\pi}{2} \int_{\frac{1}{a}}^a \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\pi}{2} \cdot \frac{2}{\sqrt{3}} \arctan \frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}} \Bigg|_{\frac{1}{a}}^a = \frac{\pi}{\sqrt{3}} \cdot \arctan \frac{2x - 1}{\sqrt{3}} \Bigg|_{\frac{1}{a}}^a = \\
 &= \frac{\pi}{\sqrt{3}} \cdot \left(\arctan \frac{2a - 1}{\sqrt{3}} - \arctan \frac{\frac{2}{a} - 1}{\sqrt{3}} \right).
 \end{aligned}$$

Answer:

$$I = \frac{\pi}{2\sqrt{3}} \cdot \left(\arctan \frac{2a - 1}{\sqrt{3}} - \arctan \frac{\frac{2}{a} - 1}{\sqrt{3}} \right).$$

Example 7. Evaluate the following integral:

$$I = \int_0^1 \ln(x + \sqrt{1 + x^2}) dx.$$

Solving. Clues:

1. Use Integration by parts.

Or

1. Calculate the inverse function.
2. Look at the lines $x = \ln(1 + \sqrt{2})$ and $y = \ln(1 + \sqrt{2})$.

Full proof. We will provide 2 proofs.

First: We will use Integration by parts,

$$\int u dv = uv - \int v du.$$

Let's substitute:

$$dv = dx, \quad u = \ln(x + \sqrt{1 + x^2}),$$

↓

$$v = x,$$

$$du = \frac{1}{x + \sqrt{1+x^2}} \cdot \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right) dx = \frac{1}{\sqrt{1+x^2}} dx.$$

$$\int \ln(x + \sqrt{1+x^2}) dx = x \cdot \ln(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} dx,$$

$$\int \ln(x + \sqrt{1+x^2}) dx = x \cdot \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

Now we can find I :

$$I = \int_0^1 \ln(x + \sqrt{1+x^2}) dx = \left(x \cdot \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} \right) \Big|_0^1 =$$

$$= \ln(1 + \sqrt{2}) - \sqrt{2} + 1.$$

Second:

$$y = f(x) = \ln(x + \sqrt{1+x^2}).$$

We can see that:

$$f'(x) > 0.$$

Therefore, the function $f(x)$ is ascending throughout its entire domain, and there exists a function $f^{-1}(x)$:

Let's build $f^{-1}(x)$:

$$y = \ln(x + \sqrt{1+x^2}) \quad (\text{blue});$$

$$x = \ln(y + \sqrt{1+y^2}) \quad (\text{red});$$

$$y - e^x = \sqrt{1+y^2} \rightarrow y - e^x \geq 0;$$

$$y^2 - 2ye^x + e^{2x} = 1 + y^2.$$

We get:

$$y = f^{-1}(x) = \frac{e^x - e^{-x}}{2} < e^x \quad (\text{red}).$$

It is known that the graphs of inverse functions are symmetric with respect to the line $y = x$ additionally, as shown in the diagram, the coordinates of point D are $(1, \ln(1 + \sqrt{2}))$ and the coordinates C are $(\ln(1 + \sqrt{2}), 1)$.

We want to find the dotted area which is equal to the dashed area (Figure 2).

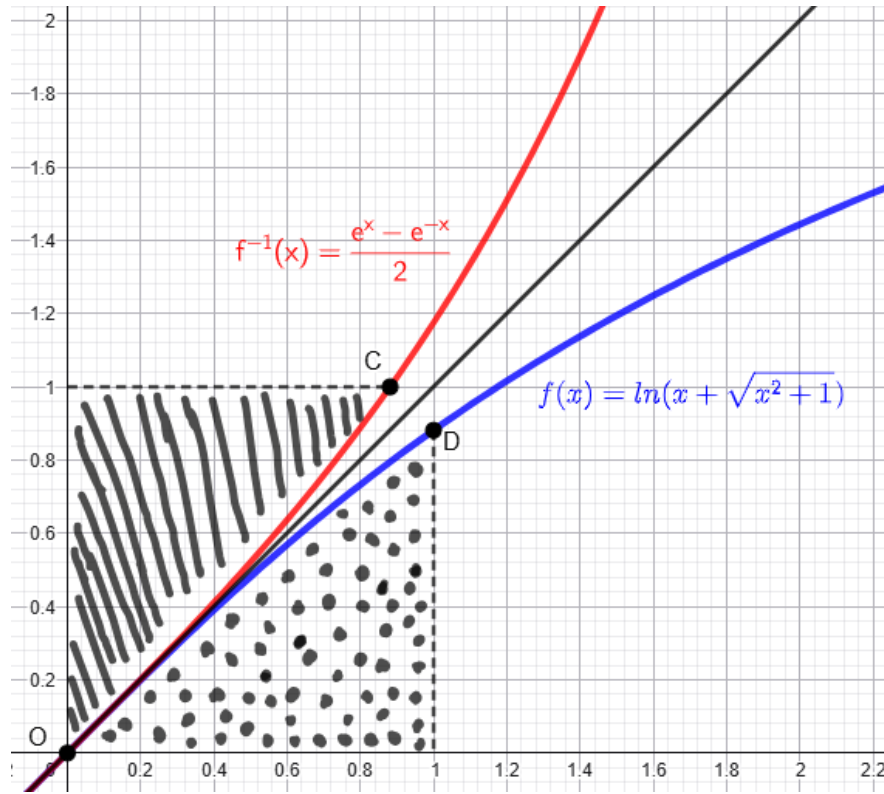


Figure 2. Diagram of $f(x)$ and its inverse

$$I = \int_0^1 \ln(x + \sqrt{1 + x^2}) dx =$$

$$= \int_0^{\ln(1+\sqrt{2})} \left(1 - \frac{e^x - e^{-x}}{2}\right) dx = \left(x - \frac{e^x - e^{-x}}{2}\right) \Big|_0^{\ln(1+\sqrt{2})} = \ln(1 + \sqrt{2}) - \sqrt{2} + 1.$$

Answer. $I = \ln(1 + \sqrt{2}) - \sqrt{2} + 1.$

4. Additional Examples (Without Solutions)

We now present several additional integrals that can be solved using the properties discussed above, but we leave their full solutions to the reader:

$$I_I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{e^{\frac{1}{x}} + 1} dx;$$

$$I_{II} = \int_0^1 \frac{\ln(1+x)}{x^2+1} dx;$$

$$I_{III} = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \tan^{\sqrt{2}} x}.$$

I_I – Cluse:

1. Use **Property (4)**.
2. Prove that:

$$I_I = \int_0^{\frac{\pi}{2}} \cos(x) \cdot \left(\frac{1}{e^{\frac{1}{x}} + 1} + \frac{1}{e^{-\frac{1}{x}} + 1} \right) dx.$$

Show that:

$$\left(\frac{1}{e^{\frac{1}{x}} + 1} + \frac{1}{e^{-\frac{1}{x}} + 1} \right) = 1.$$

Conclude that:

$$I_I = 1.$$

I_{II} – Cluse:

1. Take $x = \tan(\phi)$ and check if:

$$I_{II} = \int_0^{\frac{\pi}{4}} \ln(1 + \tan(\phi)) d\phi.$$

For $0 \leq \phi \leq \frac{\pi}{4}$.

2. Check if

$$I_{II} = \int_0^{\frac{\pi}{4}} \ln\left(1 + \tan\left(\frac{\pi}{4} - \phi\right)\right) d\phi.$$

3. Prove that:

$$\int_0^{\frac{\pi}{4}} \ln\left(1 + \tan\left(\frac{\pi}{4} - \phi\right)\right) d\phi = \int_0^{\frac{\pi}{4}} \ln(2) d\phi - \int_0^{\frac{\pi}{4}} \ln(1 + \tan(\phi)) d\phi.$$

4. Show that:

$$I_{II} = \frac{\pi}{8} \ln(2).$$

I_{III} – Cluse:

1. Define $f(x) = \frac{1}{1+\tan^{\sqrt{2}}x}$ and $g(x) = \frac{\tan^{\sqrt{2}}x}{1+\tan^{\sqrt{2}}x}$ and see that $f(x) + g(x) = 1$.
2. Prove that $g(x) = \frac{1}{1+\cotan^{\sqrt{2}}(x)}$.
3. Verify that $g(x) = f\left(\frac{\pi}{2} - x\right)$.
4. Show that:

$$\int_0^{\frac{\pi}{2}} f(x) dx + \int_0^{\frac{\pi}{2}} g(x) dx = \frac{\pi}{2}.$$

5. Prove that:

$$\int_0^{\frac{\pi}{2}} f(x) dx = \int_0^{\frac{\pi}{2}} g(x) dx.$$

6. Finally show that:

$$I_{III} = \frac{\pi}{4}.$$

5. Conclusion

In this article, we presented several properties of the definite integral that allow for the evaluation of integrals which, at first glance, seem unsolvable. The methods demonstrated here provide valuable material for teachers who wish to enrich their instruction on integration, and they also offer additional tools to students or undergraduates studying the topic of the definite integral.

In our view, both the derivation of the formulas presented in the article and the evaluation of integrals using them can serve as a source for “short questions” in high school mathematics curriculum.

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**ІНТЕГРАЛИ З ПРИХОВАНИМ РОЗВ'ЯЗКОМ: НЕМАЄ ПОТРЕБИ В СКЛАДНИХ
ОБЧИСЛЕННЯХ**

Анотація. У цій статті представлено колекцію визначених інтегралів, які, хоча, здавалося б, виходять за межі досяжності стандартних шкільних методів, можна обчислити за допомогою елементарних властивостей та формули Ньютона-Лейбніца. Ці приклади розроблені для того, щоб змінити сприйняття учнями інтегрування з суто процедурного завдання на концептуальне дослідження. Методи та ідеї, що обговорюються тут, мають на меті збагатити навчання в класі та слугувати компактними, узгодженими з навчальною програмою складними задачами.

Ключові слова: визначені інтеграли; математична освіта; концептуальне дослідження; складні задачі.

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